Topology optimization of large-scale arch systems

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1. Introduction

The topology optimization problem for arch systems (archgrids) was first put forward by G.I.N. Rozvany and W. Prager in [6] and further studied in e.g. [7, 8, 9]. The Authors discussed the optimality conditions for a structure composed of plane arches, pinned at the boundary Γ of a given plane region Ω, and transmitting to that boundary a load of given intensity $q(x, y)$, where $(x, y) \in \Omega$. Such an arch system is best visualized as a ribbed vault (arch-like roof) spanning Ω. The optimization problem is to create a structure whose weight is minimum possible, while assuming that: *i*) stresses in the entire system are only compressional; *ii*) axes of all arches belong to the same surface $f = f(x, y)$, referred to as the archgrid elevation function; *iii*) each single axis belongs to a plane perpendicular to region Ω. Arches in the vault form of a dense grid of curved bars carrying the load independently of one another. Therefore, the mechanics of a Rozvany-Prager archgrid is that of a gridwork shell and not a shell continuum.

Optimality conditions for Rozvany-Prager arch systems are compatible with those for Michell frames but with univalent – in our case compressional – stresses. Consequently, we say that optimal arch system is at the verge of a plastic failure, with each single arch uniformly compressed to the limit value, say $\sigma_C > 0$. This, in turn, involves an implicit requirement that members of optimally designed archgrid are bending- and shear-free. In other words, external load is carried by arches subjected to axial stress resultants only. Despite obvious similarities between the two theories, accommodating the Rozvany-Prager approach in computational algorithms for Michell structures is not straightforward. Loosely speaking, the main difficulty is in redefining the optimality conditions from point-wise (Michell) to arch-wise (Rozvany-Prager).

2. Computational procedure for archgrid optimization

Modern approach to archgrid optimization problem, see [2, 3, 4, 5], involves mathematical techniques of calculus of variations, thus paving way for numerical procedures in the continuous and discrete settings. In this note, we follow the latter. Numerics of the discrete approach to Rozvany-Prager archgrids is considered from the perspective based on Second-Order Cone Programming (SOCP), see [1] for theoretical introduction to this method. Procedures used for solving the examples are coded in MATLAB combined with MOSEK optimization toolbox for SOCP routines. Computational algorithm used in this outlook proves to be very efficient in terms of CPU-time. It allows for analyzing the structures with a very large ($\sim 10^6$) number of arches. Results obtained for such extremely populated archgrids clearly exceed the rational needs of civil engineering industry, but they may well serve to hint the research aimed at benchmark solutions to optimization problems. Numerical simulations for this note were performed on a laptop computer equipped with the Intel Core i7-4600U CPU @ 2.10

GHz (2 processors), 8 GB RAM, 64-bit Windows 10 Pro, MathWorks MATLAB R2021a and MOSEK optimization toolbox version 9.2.

It turns out, that the minimum volume, V_{min}, of a vault is proportional to sup $< q; f >$, with "sup" operation taken over all functions $f = f(x, y)$ satisfying the kinematic constraints imposed in the theory of Rozvany and Prager. Technically, these constraints are: *i*) $f = 0$ at Γ, and *ii*) $s_K = \nabla f \cdot \boldsymbol{e}_K$ ($K = 1, \dots, k$). Here $\mathbf{e}_K$ stands for the direction of the axis x_K orienting the K-th arch with respect to the coordinate system $(x, y) \in \Omega$, see Fig. 1, and k denotes the total number of arches in the archgrid. More precisely, the K-th arch is considered optimal if the Euclidean norm of the slope function s_K satisfies

$$\|s_K\|_2 = \sqrt{L_K}\,, \qquad \|s_K\|_2 = \left(\int_0^{L_K} \left(s_K(x_K)\right)^2 dx_K \right)^{\frac{1}{2}}, \tag{1}$$

where L_K is the length of the K-th arch chord. The requirement in (1) is known in the literature as the *Rozvany-Prager mean squared slope condition*.

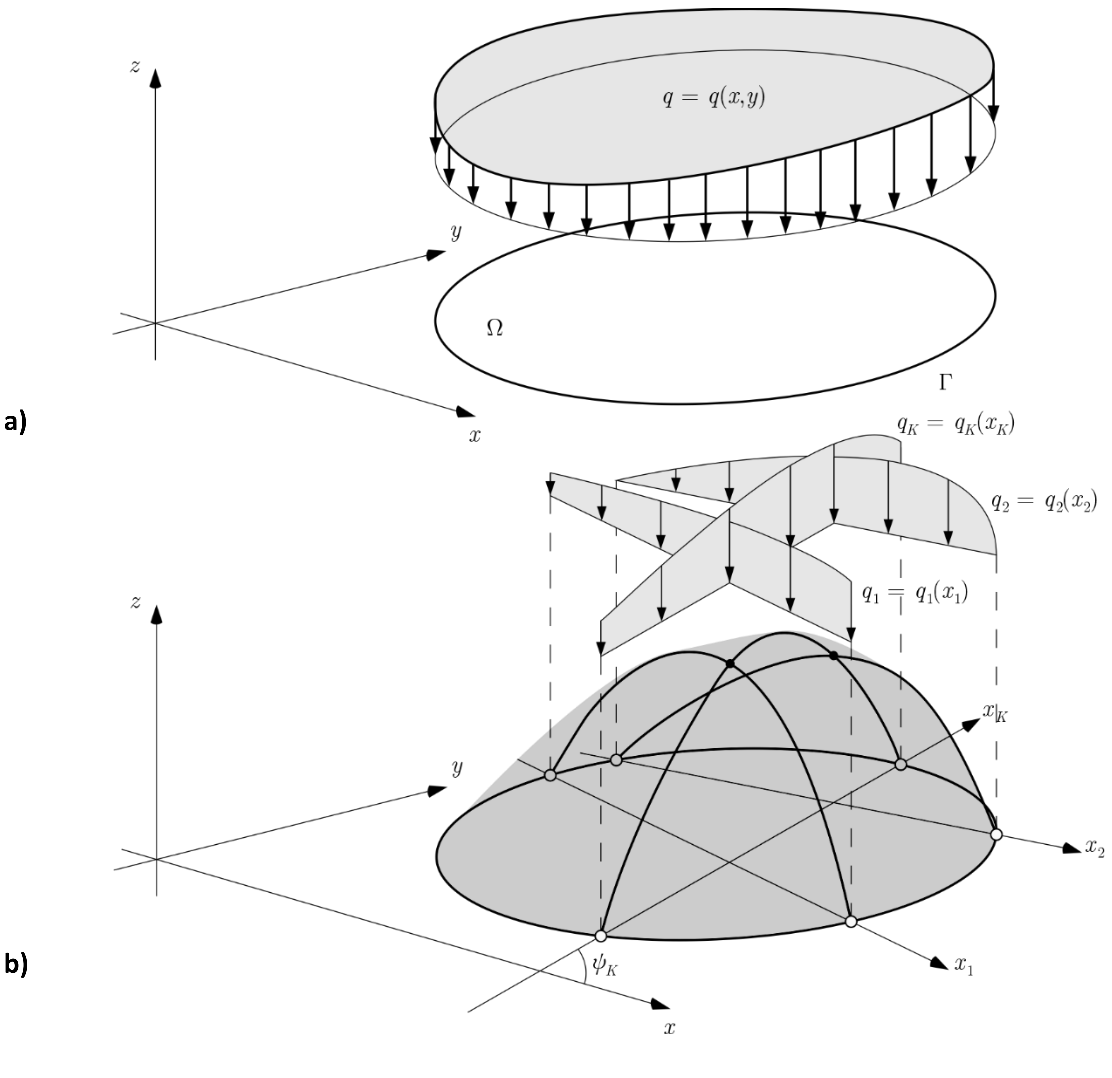

Figure 1. a) load $q = q(x, y)$ over domain Ω with boundary Γ; **b)** archgrid elevation function $f = f(x, y)$ shown in transparent grey, single arch elevation functions are shown as thick black lines, q_K represents arch load acting on the K-th arch.

In the finite-dimensional context, we choose a mesh of n nodes in Ω at which the elevation of surface f is sampled, and we write $\mathbf{f} \in \mathbb{R}^n$ for the vector of elevations and $\mathbf{q} \in \mathbb{R}^n$ for the load vector. Slope functions s_K are replaced by vectors $\mathbf{s}_K = \mathbf{B}_K\mathbf{f}$, with matrices $\mathbf{B}_K$ ($K = 1, \dots, k$) taking the place of the gradient and projection operations in *ii*) above. This allows for categorizing the Rozvany-Prager minimum volume problem in terms of Second-Order Cone Programming, see [4],

(P)

$$V_{min} = \frac{2}{\sigma_C} \max\left\{\mathbf{q}^T\mathbf{f} \;\middle|\; \mathbf{f} \in \mathcal{F}\right\},$$

where:

$$\mathcal{F} = \left\{ \mathbf{f} \;\middle|\; \begin{array}{l} \mathbf{f} \in \mathbb{R}^n \text{ and } f_N = 0 \text{ if } N\text{-th node is placed at } \Gamma\,; \\ \left(\sqrt{L_K}, \mathbf{B}_K\mathbf{f}\right) \in \mathcal{C}\,. \end{array} \right\}.$$

Here, $\mathcal{C}$ stands for the second-order (quadratic) cone, i.e.

$$\mathcal{C} = \left\{(a, \mathbf{b}) \;\middle|\; a \geq \|\mathbf{b}\|_2\right\}, \tag{2}$$

where a is a positive scalar and $\|\mathbf{b}\|_2$ now stands for the Euclidean norm of vector $\mathbf{b}$. Optimization problem dual to (P) is typically introduced by the use of Lagrange multiplier technique, see [1]. Determining the values of dual variables is standard in MOSEK; we do not elaborate on this topic here for the reason of space. Let us only mention that having the dual variables, say $(\mathbf{T}_1, \dots, \mathbf{T}_k)$, one may calculate the vectors $(\mathbf{A}_1, \dots, \mathbf{A}_k)$ comprising the values of a step-function representing the varying cross-section area of each arch.

3. Example

In the example, we consider the $\Omega = [0,2L] \times [0,2L]$ and the load case $q = \text{const.}$ acting above entire Ω, see Fig. 2. Results obtained with help of the SOCP approach are compared with the conclusions from the study in [3], where the minimum volume problem was solved in the continuous setting; see Tab. 1 for the collected results. More precisely, the elevation function f has been approximated in terms of the Fourier (trigonometric) and Legendre (polynomial) series in the orthogonal coordinates. For this reason, the comparison of results is limited to the orthogonal layout of arches.

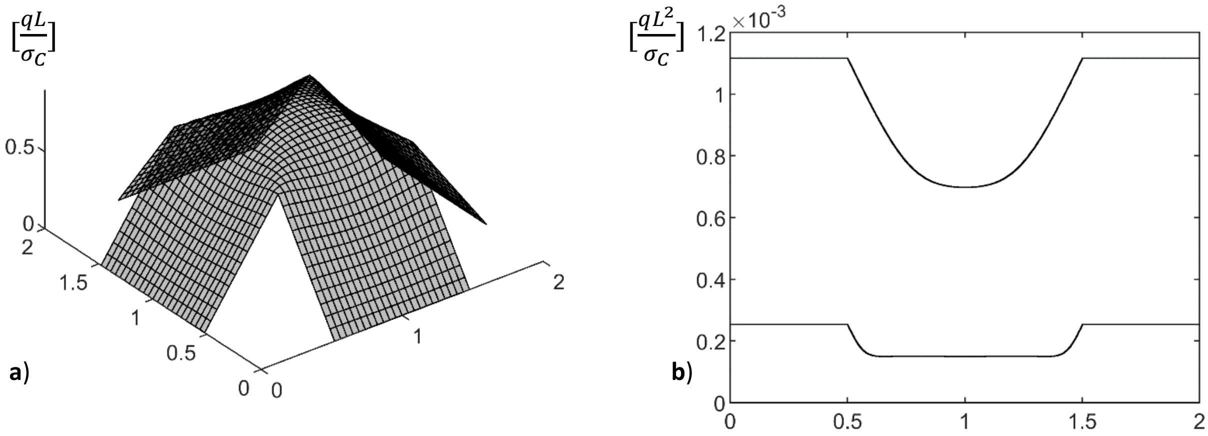

Figure 2. **a**) Optimal archgrid elevation function f; **b**) optimal cross-section area functions for arches at $x = 0.6L$ (bottom line), and $x = L$ (top line).

Table 1. Square domain Ω with $q = \mathrm{const.}$ above the entire domain. Comparison of results obtained for different numerical approaches.

number of arches along x and y	discrete approach with SOCP, [this note]		continuous approach with Fourier approximation, [3]		continuous approach with Legendre approximation, [3]	
	CPU time	optimal volume	CPU time	optimal volume	CPU time	optimal volume
45	0.5 sec.	$3.677\ \frac{qL^3}{\sigma_0}$	131 sec.	$3.681\ \frac{qL^3}{\sigma_0}$	167 sec.	$3.681\ \frac{qL^3}{\sigma_0}$
100	1.7 sec.	$3.680\ \frac{qL^3}{\sigma_0}$	4784 sec.	$3.681\ \frac{qL^3}{\sigma_0}$	9063 sec.	$3.681\ \frac{qL^3}{\sigma_0}$
1000	216 sec.	$3.681\ \frac{qL^3}{\sigma_0}$	out of memory	-	out of memory	-

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